Multiple Integrals and Vector Calculus: Synopsis

Hilary Term 2009: 14 lectures. Steve Rawlings.

1. Vectors - recap of basic principles. Things which are (and are not) vectors. Differentiation and integration of vectors. Surface area as a vector. Scalar and vector fields.
2. Gradient and the Grad (Del) operator. Evaluation in Cartesian coordinates. Examples in 2D and 3D, including methods for finding tangent planes.
3. 1D Integration - recap of basic principles. Double (2D) integrals in Cartesian coordinates. 2D integrals in plane polar coordinates. Triple (3D) integrals in cylindrical polar and spherical polar coordinate systems.
4. Line integrals. Path dependence and independence. Conservative and non-conservative fields. Potentials and state functions.
5. Jacobians. Evaluation of line and surface (2D) integrals by change of variables. 3D Jacobians.
6. Surface integrals. Evaluation of surface integrals by projection.
7. Non-conservative fields and flux integrals. Integrating factors.
8. Divergence and the Div operator. Definition. Evaluation in Cartesian coordinates. Examples including those involving lines of force.
9. Divergence Theorem. Examples including Gauss's theorem.
10. Curl. Definition. Evaluation in Cartesian coordinates. Examples including those involving rotation.
11. Stokes' theorem. Examples including Ampere's Law.
12. Second-order vector operators. Examples including a derivation of the Uniqueness Theorem.
13. Advanced Worked Examples. Including examples from Problem Set 4.
14. Advanced Topics. Suffix notation. Einstein summation convention.

The basic material will be covered in the first twelve of these lectures. Worked examples and demonstrations will be used to illustrate the key ideas throughout, but the thirteenth lecture will focus on more advanced worked examples. The fourteenth lecture will be used to introduce some more advanced topics which are not on this year's syllabus, but may be of use and interest. There are four sets of problems, each of which could be the subject of a tutorial or a class. A few problems in the final set, marked with asterisks, are based on material outside the syllabus.

Problem Set 1

1. Which of the following can be described by vectors: (a) temperature; (b) magnetic field; (c) acceleration; (d) force; (e) molecular weight; (f) area; (g) angle of polarization.

2. (a) For the following integrals sketch the region of integration and so write equivalent integrals with the order of integration reversed. Evaluate the integrals both ways.

$$\int_0^{\sqrt{2}} \int_{y^2}^{2} y \, dx \, dy \; , \qquad \int_0^4 \int_0^{\sqrt{x}} y\sqrt{x} \, dy \, dx \; , \qquad \int_0^1 \int_{-y}^{y^2} x \, dx \, dy \; .$$

(b) Reverse the order of integration and hence evaluate:

$$\int_0^{\pi} \int_y^{\pi} x^{-1} \sin x \, \mathrm{dx} \, \mathrm{dy} \; .$$

3. Find $\vec{\nabla}\phi$ in the cases: (a) $\phi = \ln|\vec{r}|$; (b) $\phi = r^{-1}$, where $r = |\vec{r}|$.

4. If $F = x^2 z + e^{y/x}$ and $G = 2z^2 y - xy^2$, find $\vec{\nabla}(F + G)$ and $\vec{\nabla}(FG)$ at $(1, 0, -2)$.

5. Two circles have equations: (i) $x^2+y^2+2ax+2by+c = 0$; and (ii) $x^2+y^2+2a'x+2b'y+c' = 0$. Show that these circles are orthogonal if $2aa' + 2bb' = c + c'$.

6. Find the equation for the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at $(1, -1, 2)$.

7. (a) A mass distribution in the positive x region of the xy-plane and in the shape of a semi-circle of radius a, centred on the origin, has mass per unit area k. Find, using plane polar coordinates,

(i) its mass M, (ii) the coordinates $(\overline{x}, \overline{y})$ of its centre of mass, (iii) its moments of inertia about the x and y axes.

(b) Do as above for a semi-infinite sheet with mass per unit area

$$\sigma = k \exp -(x^2 + y^2)/a^2 \quad \text{for} \quad x \geq 0 \; , \qquad \sigma = 0 \quad \text{for} \quad x < 0 \; .$$

where a is a constant. Comment on the comparisons between the two sets of answers.

Note that

$$\int_0^{\infty} \exp\left(-\lambda u^2\right) \mathrm{d}u = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} \; .$$

(c) Evaluate the following integral:

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2) \; \arctan(y/x) \, \mathrm{d}x \, \mathrm{d}y \; .$$

Problem Set 2

1. The pair of variables (x, y) are each functions of the pair of variables (u, v) and *vice versa*. Consider the matrices

$$A = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} .$$

(a) Show using the chain rule that the product AB of these two matrices equals the unit matrix I.

(b) Verify this property explicitly for the case in which (x, y) are Cartesian coordinates and u and v are the polar coordinates (r, θ).

(c) Assuming the result that the determinant of a matrix and the determinant of its inverse are reciprocals, deduce the relation between the Jacobians

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad \text{and} \qquad \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} .$$

2. (a) Using the change of variable $x + y = u$, $x - y = v$ evaluate the double integral $\iint_R (x^2 + y^2)\, \mathrm{d}x\mathrm{d}y$, where R is the region bounded by the straight lines $y = x$, $y = x + 2$, $y = -x$ and $y = -x + 2$.

(b) Given that $u = xy$ and $v = y/x$, show that $\partial(u, v)/\partial(x, y) = 2y/x$. Hence evaluate the integral

$$\iint \exp(-xy)\, \mathrm{d}x\mathrm{d}y$$

over the region $x > 0$, $y > 0$, $xy < 1$, $1/2 < y/x < 2$.

3. A vector field $\vec{A}(\vec{r})$ is defined by its components

$$(4x - y^4, -4xy^3 - 3y^2, 4).$$

Evaluate the line integral $\int \vec{A}.\mathrm{d}\vec{l}$ between the points with position vectors $(0,0,0)$ and $(1,2,0)$ along the following paths

(a) the straight line from $(0,0,0)$ to $(1,2,0)$;

(b) on the path of straight lines joining $(0,0,0)$, $(0,0,1)$, $(1,0,1)$, $(1,2,1)$ and $(1,2,0)$ in turn.

Show that $\vec{A}$ is conservative and find a scalar function $V(\vec{r})$ such that $\vec{A} = \vec{\nabla}(V)$.

4. A vector field $\vec{A}(\vec{r})$ is defined by its components

$$(3x^2 + 6y, -14yz, 20xz^2).$$

Evaluate the line integral $\int \vec{A}.\mathrm{d}\vec{l}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths

(a) $x = t$, $y = t^2$, $z = t^3$;

(b) on the path of straight lines joining $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, and $(1,1,1)$ in turn;

(c) the straight line joining the two points.

Is $\vec{A}$ conservative?

5. The thermodynamic relation $\delta q = C_{\mathrm{V}}\,\mathrm{d}T + (RT/V)\,\mathrm{d}V$ is not an exact differential. Show that by dividing this equation by T, it becomes exact.

6. Show that the surface area of the curved portion of a hemisphere of radius a is $2\pi a^2$ by

(i) directly integrating the element of area $a^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ over the surface of the hemisphere.

(ii) projecting onto an integral taken over the $x - y$ plane.

7. (a) Find the area of the plane $x - 2y + 5z = 13$ cut out by the cylinder $x^2 + y^2 = 9$.

(b) A uniform lamina is made of that part of the plane $x + y + z = 1$ which lies in the first octant. Find by integration its area and also its centre of mass. Use geometrical arguments to check your result for the area.

Problem Set 3

1. Spherical polar coordinates are defined in the usual way. Show that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 \sin\theta \ .$$

2. A solid hemisphere of uniform density k occupies the volume $x^2 + y^2 + z^2 \leq a^2$, $z \geq 0$. Using symmetry arguments wherever possible, find

   (i) its total mass M, (ii) the position $(\overline{x}, \overline{y}, \overline{z})$ of its centre-of-mass, and (iii) its moments and products of inertia, $I_{xx},\ I_{yy},\ I_{zz},\ I_{xy},\ I_{yz},\ I_{zx}$, where

$$I_{zz} = \int k\ (x^2 + y^2)\ \mathrm{d}V, \qquad I_{xy} = \int k\ xy\ \mathrm{d}V, \qquad \text{etc.}$$

3. Air is flowing with a speed 0.4 m s^{-1} in the direction of the vector $(-1,-1,1)$. Calculate the volume of air flowing per second through the loop which consists of straight lines joining, in turn, the following: $(1,1,0)$, $(1,0,0)$, $(0,0,0)$, $(0,1,1)$, $(1,1,1)$ and $(1,1,0)$.

4. If $\vec{n}$ is the unit normal to the surface S, evaluate $\int\int \vec{r}.\vec{n}\,\mathrm{d}S$ over (a) the unit cube bounded by the coordinate planes and the planes $x = 1$, $y = 1$ and $z = 1$; (b) the surface of a sphere of radius a centred at the origin.

5. Evaluate $\int \vec{A}.\vec{n}\ \mathrm{d}S$ for the following cases:

   - $\vec{A} = (y, 2x, -z)$ and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$.
   - $\vec{A} = (x + y^2, -2x, 2yz)$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant
   - $\vec{A} = (6z, 2x + y, -x)$ and S is the entire surface of the region bounded by the cylinder $x^2 + z^2 = 9$, $x = 0$, $y = 0$, $z = 0$ and $y = 8$.

6. The vector $\vec{A}$ is a function of position $\vec{r} = (x,y,z)$ and has components (xy^2, x^2, yz). Calculate the surface integral $\int \vec{A}.\mathrm{d}\vec{S}$ over each face of the triangular prism bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$ and $z = 1$. Show that the integral $\int \vec{A}.\mathrm{d}\vec{S}$ taken outwards over the whole surface is not zero. Show that it equals $\int \vec{\nabla}.\vec{A}\,\mathrm{d}V$ calculated over the volume of the prism. Why?

7. If $\vec{A} = (3xyz^2, 2xy^3, -x^2yz)$ and $\phi = 3x^2 - yz$, find: (a) $\vec{\nabla}.\vec{A}$; (b) $\vec{A}.\vec{\nabla}\phi$; (c) $\vec{\nabla}.(\phi\vec{A})$; (d) $\vec{\nabla}.(\vec{\nabla}\phi)$.

8. The magnetic field $\vec{B}$ at a distance r from a straight wire carrying a current I has magnitude $\mu_0 I/2\pi r$. The lines of force are circles centred on the wire and in planes perpendicular to it. Show that $\vec{\nabla}.\vec{B} = 0$.

Problem Set 4

1. O is the origin and A, B, C are points with position vectors $\vec{a} = (1, 0, 0)$, $\vec{b} = (1, 1, 1)$ and $\vec{c} = (0, 2, 0)$ respectively. Find the vector area $\vec{S}$ of the loop OABCO (a) by drawing the loop in projection onto the yz, zx and xy planes and calculating the components of $\vec{S}$, and (b) by filling the loop with (e.g. 2 or 3) plane polygons, ascribing a vector area to each and taking the resultant. Calculate the projected area of the loop (i) when seen from the direction which makes it appear as large as possible, and (ii) when seen from the direction of the vector $(0, -1, 1)$? What are the corresponding answers for the loop OACBO?

2. Calculate the solid angle of a cone of half-angle α.

3. A body expands linearly by a factor α because of a rise in temperature. Because of the expansion, a point at position $\vec{r}$ is displaced to $\vec{r} + \vec{h}$. Calculate $\vec{\nabla}.\vec{h}$. By what fraction does the volume increase?

4. Sketch the vector fields $\vec{A} = (x, y, 0)$ and $\vec{B} = (y, -x, 0)$. Calculate the divergence and curl of each vector field and explain the physical significance of the results obtained.

5. The vector $\vec{A}(\vec{r}) = (y, -x, z)$. Verify Stokes' Theorem for the hemispherical surface $|\vec{r}| = 1$, $z \geq 0$.

6. $\vec{A} = (y, -x, 0)$. Find $\int \vec{A}.\vec{dl}$ for a closed loop on the surface of the cylinder $(x-3)^2 + y^2 = 2$.

7. A bucket of water is rotated slowly with angular velocity ω about its vertical axis. When a steady state has been reached the water rotates with a velocity field $\vec{v}(\vec{r})$ as if it were a rigid body. Calculate $\vec{\nabla}.\vec{v}$ and interpret the result. Calculate $\vec{\nabla} \times \vec{v}$. Can the flow be represented in terms of a velocity potential ϕ such that $\vec{v} = \vec{\nabla}\phi$? If so, what is ϕ?

8. Prove: (a) $\vec{\nabla} \times (\vec{\nabla}\phi) = 0$; and (b) $\vec{\nabla}.(\vec{\nabla} \times \vec{A}) = 0$.

9. If $\phi = 2xyz^2$, $\vec{F} = (xy, -z, x^2)$ and C is the curve $x = t^2$, $y = 2t$,$z = t^3$ from $t = 0$ to $t = 1$, evaluate the line integrals: (a) $\int_C \phi \, \mathrm{d}\vec{r}$; (b) $\int_C \vec{F} \times \mathrm{d}\vec{r}$.

10. * To what scalar or vector quantities do the following expressions in suffix notation correspond (expand and sum where possible): $a_i b_j c_i$; $a_i b_j c_j d_i$; $\delta_{ij} a_i a_j$; $\delta_{ij}\delta_{ij}$; $\epsilon_{ijk} a_i b_k$; and $\epsilon_{ijk}\delta_{ij}$.

11. * Use summation convention to find the grad of the following scalar functions of position $\vec{r} = (x, y, z)$: (a) $|r|^n$, (b) $\vec{a}.\vec{r}$.

12. * Use summation convention to find the div and curl of the following vector functions of position $\vec{r} = (x, y, z)$: (a) r; (b) $|\vec{r}|^n \vec{r}$; (c) $(\vec{a}.\vec{r})\vec{b}$; and (d) $\vec{a} \times \vec{r}$. Here, $\vec{a}$ and $\vec{b}$ are fixed vectors.

13. * Use summation convention to prove: (a) $\vec{\nabla} \times (\vec{\nabla}\phi) = 0$; and (b) $\vec{\nabla}.(\vec{\nabla} \times \vec{A}) = 0$ (i.e. repeat question 5 above but hopefully by a faster method).